

# The Ubiquity of Sidon sets that are not $I_0$

Kathryn E. Hare and L. Thomas Ramsey

**ABSTRACT.** We prove that every infinite, discrete abelian group admits a pair of  $I_0$  sets whose union is not  $I_0$ . In particular, this implies that every such group contains a Sidon set that is not  $I_0$ .

## 1. Introduction

A subset  $E$  of a discrete abelian group  $\Gamma$  with compact dual group  $G$  is said to be a Sidon set if every bounded  $E$ -sequence can be interpolated by the Fourier transform of a measure on  $G$ . If the measure can be chosen to be discrete,  $E$  is called an  $I_0$  set. Finite sets in any group  $\Gamma$  and Hadamard sequences of integers are examples of these interpolation sets in the group  $\Gamma = \mathbb{Z}$ .

Clearly,  $I_0$  sets are Sidon. However, the converse is not true since the class of Sidon sets is known to be closed under finite unions, but the class of  $I_0$  sets is not. Indeed, in [12], Méla gave an example of a pair of Hadamard sets in  $\mathbb{Z}$  whose union is not  $I_0$ .

In this note we prove that every infinite, discrete abelian group admits a pair of  $I_0$  sets whose union is not  $I_0$ . Consequently, every such group admits a Sidon set that is not  $I_0$ . Our method is constructive and establishes even more: We prove that given any infinite subset  $F \subseteq \Gamma$  there are  $I_0$  sets  $E \subseteq F$  and  $E' \subseteq F + F - F$ , whose union is not  $I_0$  (but is, of course, Sidon). In fact, we show that the sets  $E, E'$  have stronger interpolation properties than just  $I_0$ . These depend upon the algebraic properties of the initial set  $F$ .

## 2. Preliminaries

Let  $G$  be a compact abelian group and  $\Gamma$  its discrete abelian dual group. One example is  $G = \mathbb{T}$ , the circle group of complex numbers of modulus one, with dual group  $\Gamma = \mathbb{Z}$ .

**DEFINITION 1.** (i) A subset  $E \subseteq \Gamma$  is said to be **Sidon** if for every bounded function  $\phi : E \rightarrow \mathbb{C}$  there is a measure  $\mu$  on  $G$  with  $\widehat{\mu}(\gamma) = \phi(\gamma)$  for all  $\gamma \in E$ . If the interpolating measure  $\mu$  can always be chosen to be discrete, then the set  $E$  is said to be  $I_0$ .

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(ii) A subset  $E \subseteq \Gamma$  is said to be  $\varepsilon$ -**Kronecker** if for every  $\phi : E \rightarrow \mathbb{T}$  there exists  $x \in G$  such that

$$(2.1) \quad |\phi(\gamma) - \gamma(x)| < \varepsilon \text{ for all } \gamma \in E$$

and is called **weak**  $\varepsilon$ -**Kronecker** if the strict inequality is replaced by  $\leq$ .

Hadamard sets  $E = \{n_j\} \subseteq \mathbb{N}$  with Hadamard ratio  $q = \inf n_{j+1}/n_j > 1$  are  $I_0$  sets [11], thus both  $\{3^j\}_{j \geq 1}$  and  $\{3^j + j\}_{j \geq 1}$  are  $I_0$  subsets of  $\mathbb{Z}$ . However, Méla, in [12], proved that their union is not  $I_0$ . This shows that not all Sidon sets in  $\mathbb{Z}$  are  $I_0$  since it is a deep result, first obtained by Drury [1], with a later proof given by Pisier [13], that any finite union of Sidon sets (in any group  $\Gamma$ ) is Sidon. It is an open problem whether every Sidon set is a finite union of  $I_0$  sets.

Hadamard sets with ratio  $q > 2$  are also known to be weak  $\varepsilon$ -Kronecker with  $\varepsilon = |1 - e^{i\pi(q-1)}|$ . In fact, every  $\varepsilon$ -Kronecker set is  $I_0$  if  $\varepsilon < \sqrt{2}$  [5] and is Sidon if  $\varepsilon < 2$  [8].

It is known that every infinite subset of  $\Gamma$  contains infinite  $I_0$  sets and if  $\Gamma$  does not contain any elements of order 2, then every infinite subset contains an infinite weak 1-Kronecker set; c.f., [4], [6], [7] and the references cited there. Of course, if  $\Gamma$  consists of only elements of order two, then it does not contain any  $\varepsilon$ -Kronecker set with  $\varepsilon < \sqrt{2}$ .

The main result of this paper is to show that *every* infinite, discrete abelian group admits a pair of  $I_0$  sets that are weak  $\varepsilon$ -Kronecker for suitable  $\varepsilon$ , but whose union is not  $I_0$ . The number  $\varepsilon$  can always be chosen to be at most  $\sqrt{2}$  and often can be taken to be arbitrarily small, such as when  $\Gamma$  is torsion free. As far as we are aware, this is the first proof that every infinite, discrete abelian group admits Sidon sets that are not  $I_0$ . Knowing the existence of such sets is useful, for instance, in studying the space of weakly almost periodic functions on  $\Gamma$ ; c.f., [3, p. 17], as well as [2] for further background.

Like Méla's original proof, our argument is constructive and relies upon the well known Hartman/Ryll-Nardzewski characterization of  $I_0$  sets in terms of the Bohr topology on  $\overline{\Gamma}$ , the Bohr compactification of  $\Gamma$ .

**PROPOSITION 1.** [9] *A subset  $E \subseteq \Gamma$  is  $I_0$  if and only if whenever  $E_1$  and  $E_2$  are disjoint subsets of  $E$ , then  $E_1$  and  $E_2$  have disjoint closures in  $\overline{\Gamma}$ .*

With this, one can quickly prove the following generalization of Méla's argument.

**LEMMA 1.** *Suppose that 0 is a cluster point of  $\{\chi_n\}_{n=1}^\infty \subseteq \Gamma$  in the Bohr compactification of  $\Gamma$ . If the sets  $E = \{\gamma_n\} \subseteq \Gamma$  and  $E' = \{\gamma_n + \chi_n\} \subseteq \Gamma$  are disjoint, then  $E \cup E'$  is not an  $I_0$  set.*

**PROOF.** Assume the subnet  $\{\chi_{n_\alpha}\}$  (indexed by  $\alpha$ ) converges to 0 in  $\overline{\Gamma}$ . Since  $\overline{\Gamma}$  is compact, there is a subnet (not relabelled) such that  $\{\gamma_{n_\alpha}\}$  converges to some  $\gamma \in \overline{\Gamma}$ . Since  $\{\chi_{n_\alpha}\}$  also converges to 0 along this subnet, it follows that  $\{\gamma_{n_\alpha} + \chi_{n_\alpha}\}$  converges to  $\gamma$ . But that means the disjoint sets  $E$  and  $E'$  both have  $\gamma$  in their closures. By Proposition 1, their union is not  $I_0$ .  $\square$

In the case of  $\Gamma = \mathbb{Z}$ , one can take  $\{\chi_n\} = \mathbb{N}$ . For the more general situation, we will take  $\{\chi_n\} = \bigcup_{m=1}^\infty H_m$ , where the sets  $H_m$  are constructed in the following lemma.

LEMMA 2. *If  $F$  is any infinite subset of  $\Gamma$ , then there is a countable subset  $H \subseteq (F - F) \setminus \{0\}$  which has 0 as a cluster point in the Bohr topology. Indeed, we can take  $H = \bigcup_{m=1}^{\infty} H_m$  where for each positive integer  $m$ ,  $H_m$  is a finite subset of  $(F - F) \setminus \{0\}$  having the property that for all  $x_1, \dots, x_m \in G$  there is some  $\gamma \in H_m$  with*

$$\sup_{1 \leq j \leq m} |\gamma(x_j) - 1| < \frac{1}{m}.$$

PROOF. To begin, we claim that there is some  $\phi \in \overline{\Gamma}$  with the property that if  $V$  is an open neighbourhood of  $\phi$ , then  $V \cap F$  is infinite. If not, then for every  $\phi \in \overline{\Gamma}$  there is an open neighbourhood of  $\phi$ , say  $U_\phi$ , such that  $U_\phi \cap F$  is finite. As the open sets  $U_\phi$  cover  $\overline{\Gamma}$  and  $\overline{\Gamma}$  is compact, we can choose a finite subcover,  $\{U_{\phi_j}\}_{j=1}^J$ . But that contradicts the assumption that  $F$  is infinite and hence proves the claim.

In particular, if  $x_1, \dots, x_m \in G$ , then

$$V = \left\{ \gamma \in \overline{\Gamma} : \sup_{j=1, \dots, m} |\gamma(x_j) - \phi(x_j)| < \frac{1}{2m} \right\}$$

is a neighbourhood of  $\phi$  and hence must contain infinitely many elements from  $F$ . For each such collection  $x_1, \dots, x_m$ , choose  $f_1 \neq f_2$  in  $F$  such that  $|f_i(x_j) - \phi(x_j)| < 1/(2m)$  for all  $j = 1, \dots, m$ . Then  $|f_1(x_j) - f_2(x_j)| < 1/m$  for  $i = 1, 2$  and all  $j = 1, \dots, m$ . Since  $f_1 - f_2 \in F - F \subseteq \Gamma$ , and is therefore a continuous character on  $G$ , there is a neighbourhood of  $(x_1, \dots, x_m) \in G^m$ , denoted  $W(f_1, f_2)$ , such that if  $(y_1, \dots, y_m) \in W(f_1, f_2)$ , then

$$\sup_{j=1, \dots, m} |f_1(y_j) - f_2(y_j)| < \frac{1}{m}.$$

As  $G^m$  is compact, there are finitely many such neighbourhoods,  $W(f_1^{(k)}, f_2^{(k)})$ , for  $k = 1, \dots, N_m$ , so that  $G^m = \bigcup_{k=1}^{N_m} W(f_1^{(k)}, f_2^{(k)})$ . The set

$$H_m = \{f_2^{(k)} - f_1^{(k)} : k = 1, \dots, N_m\}$$

meets the requirements of the lemma.

The observation that  $H = \bigcup_{m=1}^{\infty} H_m$  clusters at 0 follows directly from the fact that any neighbourhood of 0 in the Bohr topology contains a subset of the form  $\{\gamma : \sup_{j=1, \dots, m} |\gamma(x_j) - 1| < 1/m\}$  for some positive integer  $m$  and  $x_1, \dots, x_m \in G$ .  $\square$

Before turning to the details of the proof of our main result, we list some other elementary facts about Kronecker and  $I_0$  sets which can be found in [7] and will be used in the proof of our main result.

LEMMA 3. (i) *Suppose  $\pi : \Gamma \rightarrow \Lambda$  is a homomorphism that is an injection on  $E \subseteq \Gamma$ . If  $\pi(E)$  is weak  $\varepsilon$ -Kronecker (or  $I_0$ ) as a subset of the group  $\pi(\Gamma)$ , then  $E$  is weak  $\varepsilon$ -Kronecker (resp.,  $I_0$ ) as a subset of  $\Gamma$ .*

(ii) *Suppose  $\Lambda$  is a subgroup of  $\Gamma$  and that  $E$  is weak  $\varepsilon$ -Kronecker (or  $I_0$ ) as a subset of the group  $\Lambda$ . Then  $E$  is also weak  $\varepsilon$ -Kronecker (resp.,  $I_0$ ) as a subset of  $\Gamma$ .*

### 3. The Main Result

**3.1. Statement and outline of the proof.** Here is the statement of our main result.

**THEOREM 1.** *If  $F$  is any infinite subset of a discrete abelian group  $\Gamma$ , there are countable, disjoint sets  $E \subseteq F$  and  $E' \subseteq F + F - F$  such that both  $E$  and  $E'$  are  $I_0$ , but  $E \cup E'$  is not  $I_0$ . Furthermore, the sets  $E, E'$  can be chosen to be weak  $\varepsilon$ -Kronecker for suitable  $\varepsilon \leq \sqrt{2}$ .*

**REMARK 1.** *The choice of  $\varepsilon$  will be clear from the proof and depends on algebraic properties of  $F$ . As will be seen in the proof, in many situations  $\varepsilon$  can be chosen to be arbitrarily small.*

Since the union of any two Sidon sets is again Sidon, we immediately obtain the following corollary.

**COROLLARY 1.** *Every infinite, discrete abelian group admits a Sidon set that is not  $I_0$ .*

The remainder of the paper will be devoted to proving the theorem. Its proof depends upon the general structure theory for abelian groups.

**THEOREM 2.** *(see [7, p. 165]) Given any discrete abelian group  $\Gamma$ , there is an index set  $\mathcal{I}$  such that  $\Gamma$  is isomorphic to a subgroup of*

$$\Omega = \bigoplus_{\alpha \in \mathcal{I}} \Omega_\alpha,$$

where for each  $\alpha$  either  $\Omega_\alpha = \mathbb{Q}$  or there is a prime number  $p_\alpha$  such that  $\Omega_\alpha = C(p_\alpha^\infty)$ , the group of all  $p_\alpha^n$ -th roots of unity.

Throughout the remainder of the paper  $\pi_\alpha$  will denote the projection from  $\Omega$  onto the factor group  $\Omega_\alpha$ . Our proof of Theorem 1 will be constructive and will depend on the following two cases:

Case 1: There is some index  $\alpha \in \mathcal{I}$  such that  $\pi_\alpha(F)$  is infinite. This case will be handled by Lemma 4 when  $\Omega_\alpha = \mathbb{Q}$  and Lemma 5 when  $\Omega_\alpha = C(p_\alpha^\infty)$ . We will see that we can even arrange for the sets  $E$  and  $E'$  to be  $\varepsilon$ -Kronecker for any given  $\varepsilon > 0$ . Making the choice with  $\varepsilon < \sqrt{2}$  ensures  $E, E'$  are both  $I_0$ .

Case 2: Otherwise,  $\pi_\alpha(F)$  is finite for all indices  $\alpha \in \mathcal{I}$  and then there must be an infinite subset  $\mathcal{J} \subseteq \mathcal{I}$  such that for each  $\alpha \in \mathcal{J}$  there is some  $\lambda \in F$  with  $\pi_\alpha(\lambda) \neq 0$ . The existence of such an infinite subset of indices allows us to either construct sets  $E, E'$  that are weak  $\varepsilon$ -Kronecker for some  $\varepsilon < \sqrt{2}$  (and hence  $I_0$ ) or to construct sets  $E, E'$  that are both independent (and hence  $I_0$ ) and weak  $\sqrt{2}$ -Kronecker. The choice of construction depends on the orders of the non-zero characters  $\pi_\alpha(\lambda)$ . This argument can be found in Lemma 6.

In both cases, the two sets we construct will be disjoint and have the form  $E = \{\gamma_n\}$ ,  $E' = \{\gamma_n + \chi_n\}$  where  $\{\chi_n\}$  clusters at 0. Thus the fact that  $E \cup E'$  is not  $I_0$  will follow immediately from Lemma 1.

We now turn to handling these two cases.

#### 3.2. Proof of the Theorem in Case 1.

**LEMMA 4.** *Suppose there exists an index  $\alpha \in \mathcal{I}$  such that  $\pi_\alpha(F)$  is infinite and  $\Omega_\alpha = \mathbb{Q}$ . Given any  $\varepsilon > 0$ , there are infinite disjoint sets  $E \subset F$  and  $E' \subset F + F - F$  such that  $E$  and  $E'$  are weak  $\varepsilon$ -Kronecker and  $I_0$ , but  $E \cup E'$  is not  $I_0$ .*

PROOF. Let  $H = \bigcup_{m=1}^{\infty} H_m = \{\chi_n\}_{n=1}^{\infty} \subseteq (F - F) \setminus \{0\}$  be a set that clusters at 0, constructed in Lemma 2. Fix  $0 < \varepsilon < \sqrt{2}$  and assume we can find a sequence of characters  $\lambda_n \in \pi_{\alpha}(F)$  such that:

- (a)  $V = \{\lambda_n\}$  and  $V' = \{\lambda_n + \pi_{\alpha}(\chi_n)\}$  are weak  $\varepsilon$ -Kronecker sets in  $\Omega_{\alpha}$ ; and
- (b) For  $n \neq n'$  we have  $\lambda_n \neq \lambda_{n'}$ ,  $\lambda_n \neq \lambda_{n'} + \pi_{\alpha}(\chi_{n'})$ , and  $\lambda_n + \pi_{\alpha}(\chi_n) \neq \lambda_{n'} + \pi_{\alpha}(\chi_{n'})$ .

Then, for each  $\lambda_n$  choose some  $\gamma_n \in F$  such that  $\pi_{\alpha}(\gamma_n) = \lambda_n$ . Set  $E = \{\gamma_n\} \subseteq F$  and  $E' = \{\gamma_n + \chi_n\} \subseteq F + F - F$ . By construction,  $\pi_{\alpha}$  is one-to-one from  $E$  to  $V$  and one-to-one from  $E'$  to  $V'$ . Condition (b) and the fact that  $H$  consists of nonzero characters implies that  $E$  and  $E'$  consist of distinct terms and are disjoint.

By Lemma 3(i),  $E$  and  $E'$  inherit the weak  $\varepsilon$ -Kronecker property from  $V$  and  $V'$  respectively. Since  $\varepsilon < \sqrt{2}$ , both  $E$  and  $E'$  are  $I_0$ . Furthermore, because 0 is a cluster point of the set  $H$ , Lemma 1 implies that  $E \cup E'$  is not  $I_0$ .

Thus the proof of the lemma will be complete if we can construct a sequence of characters satisfying the two conditions (a) and (b). This will be an induction argument which depends on whether  $\pi_{\alpha}(F)$  is a subset of a group isomorphic to  $\mathbb{Z}$  or it is not.

First, suppose that there is some integer bound  $B > 0$  such that for all  $\lambda \in \pi_{\alpha}(F)$ , there are integers  $b > 0$  and  $a$  such that  $\pi_{\alpha}(\lambda) = a/b$  and  $b \leq B$ . Then  $\pi_{\alpha}(F)$  is a subset of the (additive) subgroup  $\frac{1}{B!}\mathbb{Z}$  of  $\mathbb{Q}$ . Because  $H \subseteq F - F$ , we also have  $\pi_{\alpha}(H)$  contained in this subgroup.

Given  $\varepsilon > 0$ , choose an integer  $q > 2$  such that  $\pi/(q-1) < \varepsilon$ . As  $\pi_{\alpha}(F)$  is infinite, we may inductively choose  $\lambda_n \in \pi_{\alpha}(F)$ , sufficiently large in modulus, so that both  $V$  and  $V'$  are Hadamard sequences in  $\mathbb{Q}$  with Hadamard ratio  $\geq q$  and condition (b) is satisfied. By [7, Prop. 2.2.6],  $V$  and  $V'$  are both weak  $\varepsilon$ -Kronecker subsets of  $\frac{1}{B!}\mathbb{Z}$  and by Lemma 3(ii) they are also both weak  $\varepsilon$ -Kronecker sets in  $\Omega_{\alpha}$ . Thus condition (a) is satisfied.

Otherwise, for every positive integer  $B$ , there is some  $s/t \in \pi_{\alpha}(F)$  with  $t > B$  and  $\gcd(s, t) = 1$ . (We will say  $s/t$  is in reduced form.) In this case we need to carefully account for the denominators of rational numbers. Note that any  $x \in \mathbb{Q}$  has a unique reduced form,  $s/t$ , and we will write  $D(x)$  for the denominator  $t$ .

Given  $\varepsilon > 0$ , choose an integer  $q > 2$  such that  $\pi/q < \varepsilon$ . Let  $B_0 = D(\pi_{\alpha}(\chi_1))$  and choose  $\lambda_1 \in \pi_{\alpha}(F)$  such that  $D(\lambda_1) > qB_0!$ . Assuming  $\lambda_1, \dots, \lambda_n$  have been inductively constructed for  $n \geq 1$ , let

$$B_n = 2 \max\{D(\lambda_i), D(\pi_{\alpha}(\chi_j)) : 1 \leq i \leq n, 1 \leq j \leq n+1\}.$$

Now choose  $\lambda_{n+1} \in \pi_{\alpha}(F)$  so that  $D(\lambda_{n+1}) > qB_n!$ . This choice ensures that  $\lambda_i$  and  $\lambda_i + \pi_{\alpha}(\chi_i)$  for  $1 \leq i \leq n$ , as well as  $\pi_{\alpha}(\chi_{n+1})$ , all belong to  $\frac{1}{B_n!}\mathbb{Z}$ , while  $\lambda_{n+1}$  and  $\lambda_{n+1} + \pi_{\alpha}(\chi_{n+1})$  are outside  $\frac{1}{B_n!}\mathbb{Z}$ . It follows that condition (b) will be satisfied for  $V$  and  $V'$ .

We argue next that  $V'$  is  $\varepsilon$ -Kronecker in  $\Omega_{\alpha}$ . To this end, let  $\phi : V' \rightarrow \mathbb{T}$ , say  $\phi(\lambda_n + \pi_{\alpha}(\chi_n)) = t_n \in \mathbb{T}$ . We need to prove there is some character  $g \in \widehat{\mathbb{Q}}$ , the dual of  $\mathbb{Q}$ , such that

$$(3.1) \quad |g(\lambda_n + \pi_{\alpha}(\chi_n)) - t_n| < \varepsilon \text{ for all } n.$$

As explained in [10, 25.5], elements of  $\widehat{\mathbb{Q}}$  can be identified with sequences  $\{\omega_n\} \subset \mathbb{T}$ , subject to the constraints that  $\omega_{n+1}^{n+1} = \omega_n$ , with the understanding that  $g(1/n!) =$

$\omega_n$ . Clearly it will be sufficient to satisfy the consistency condition

$$(3.2) \quad \omega_{B_{n+1}}^{B_{n+1}!/B_n!} = \omega_{B_n},$$

provided that for  $j \notin \{B_n\}$ , say  $B_n < j < B_{n+1}$ , one specifies

$$\omega_j = \omega_{B_{n+1}}^{B_{n+1}!/j!}.$$

To start the specification of  $g$ , set  $\omega_{B_0} = 1$ . This ensures that if  $k \in \mathbb{Z}$ , then  $g(k) = \omega_{B_0}^{kB_0!}$ , hence  $g(\mathbb{Z}) = 1$ . (This will be helpful in the next lemma as it allows us to interpret  $g$  as a character on  $\mathbb{Q}/\mathbb{Z}$ .) Since  $\pi_\alpha(\chi_1) = s/B_0!$  for some integer  $s$ , it follows that

$$g(\pi_\alpha(\chi_1)) = \omega_{B_0}^s = 1.$$

By Equation (3.2) one may choose  $\omega_{B_1}$  to be any  $J$ -th root of unity, where  $J = B_1!/B_0!$ , in other words, one can choose any integer  $K \in [0, J-1]$  and specify

$$\omega_{B_1} = e^{2\pi i K/J}.$$

Because  $\lambda_1 \in \frac{1}{B_1!}\mathbb{Z}$ , the reduced form of  $\lambda_1$  is  $s/t$  with  $t$  dividing  $B_1!$ . Thus, with  $y/z$  the reduced form of  $B_0!/t$ , we have

$$g(\lambda_1) = \left(e^{2\pi i K/J}\right)^{sB_1!/t} = e^{2\pi i K s B_0!/t} = e^{2\pi i K s y/z}.$$

Since  $t = D(\lambda_1) > qB_0!$ , the reduced form of  $B_0!/t$  is  $y/z$  with  $z > q$ . Both  $s$  and  $y$  are relatively prime to  $z$ , hence the exponential  $e^{2\pi i s y/z}$  is a primitive  $z$ -th root of unity. By definition,  $B_1 \geq 2t$  and  $t > qB_0!$ , thus  $J = B_1!/B_0! \geq B_1 > t \geq z$ . This means we can choose any of the  $z$ -th roots of unity as the value for  $g(\lambda_1)$ . If we make a choice that is closest to  $t_1$ , then the angular difference between  $g(\lambda_1)$  and  $t_1$  is at most  $\pi/z$  and thus

$$|g(\lambda_1 + \pi_\alpha(\chi_1)) - 1| = |g(\lambda_1) - t_1| < \frac{\pi}{z} < \frac{\pi}{q} < \varepsilon.$$

We proceed to define  $\{\omega_{B_n}\}$  inductively. Assume that for  $1 \leq j \leq n$  we have specified  $\omega_{B_j}$  so that  $|g(\lambda_j + \pi_\alpha(\chi_j)) - t_j| < \varepsilon$ . By the definition of  $B_n$  we know that  $\pi_\alpha(\chi_{n+1})$  is in  $\frac{1}{B_n!}\mathbb{Z}$ . Therefore  $g$  has already been specified at  $\pi_\alpha(\chi_{n+1})$ . However,  $\lambda_{n+1} \notin \frac{1}{B_n!}\mathbb{Z}$ , but is inside  $\frac{1}{B_{n+1}!}\mathbb{Z}$ , so the selection of  $\omega_{B_{n+1}}$  will determine  $g(\lambda_{n+1})$ .

By Equation (3.2),  $\omega_{B_{n+1}}$  can be chosen to be any  $J$ -th root of  $\omega_{B_n}$ , where  $J = B_{n+1}!/B_n!$ . If we write  $e^{i\theta}$  for  $\omega_{B_n}$ , then we are free to choose any integer  $K \in [0, J-1]$  and define

$$\omega_{B_{n+1}} = e^{i\theta/J} \cdot e^{2\pi i K/J}.$$

Let  $s/t$  be the reduced form of  $\lambda_{n+1}$  and  $y/z$  the reduced form for  $B_n!/t$ . Since  $t$  divides  $B_{n+1}!$ , we have

$$g(\lambda_{n+1}) = \omega_{B_{n+1}}^{sB_{n+1}!/t} = e^{i\theta s B_n!/t} \cdot e^{2\pi i K s B_n!/t} = e^{i\theta s B_n!/t} \cdot e^{2\pi i K s y/z}.$$

As before we see that any of the  $z$ -th roots of unity can be used to help define  $g(\lambda_{n+1})$  and we make the choice (of  $K$ ) so that the corresponding  $z$ -th root of unity differs in angle by at most  $\pi/q$  from

$$\left(e^{i\theta s B_n!/t}\right)^{-1} g(\pi_\alpha(\chi_{n+1}))^{-1} t_{n+1}.$$

(We remind the reader that the first two factors above are known as they have already been determined by  $\omega_{B_n}$ .) Therefore

$$\begin{aligned} |g(\lambda_{n+1} + \pi_\alpha(\chi_{n+1})) - t_{n+1}| &= \left| e^{2\pi i Ksy/z} - \left( e^{i\theta s B_n!/t} \right)^{-1} g(\pi_\alpha \chi_{n+1})^{-1} t_{n+1} \right| \\ &< \frac{\pi}{z} < \frac{\pi}{q} < \epsilon. \end{aligned}$$

Since the choice of  $g$  satisfies (3.1), it follows that  $V'$  is  $\varepsilon$ -Kronecker.

The proof that  $V$  is  $\varepsilon$ -Kronecker is similar, but easier, as the factors  $g(\pi_\alpha(\chi_{n+1}))$  are not present. This shows that condition (a) also holds and that completes the proof of the Lemma.  $\square$

**LEMMA 5.** *Suppose there exists an index  $\alpha \in \mathcal{I}$  such that  $\pi_\alpha(F)$  is infinite and  $\Omega_\alpha = C(p^\infty)$ . Given any  $\varepsilon > 0$ , there are infinite disjoint sets  $E \subset F$  and  $E' \subset F + F - F$  such that  $E, E'$  are weak  $\varepsilon$ -Kronecker and  $I_0$ , but  $E \cup E'$  is not  $I_0$ .*

**PROOF.** Let  $H = \{\chi_n\} \subseteq (F - F) \setminus \{0\}$  be a countable set that clusters at 0, as in the previous lemma. We identify  $C(p^\infty)$  with a subgroup of  $\mathbb{Q}/\mathbb{Z}$ , so that for every  $\lambda$  in the subgroup generated by  $F$  there is some  $x_\lambda \in \mathbb{Q}$  such that  $\pi_\alpha(\lambda) = x_\lambda + \mathbb{Z}$ .

Because  $\pi_\alpha(F)$  is infinite, the set of minimal denominators  $\{D(x_\lambda) : \lambda \in F\}$  must be unbounded. The proof of the second part of Lemma 4 shows that given  $0 < \varepsilon < \sqrt{2}$  there is a sequence  $V = \{x_{\lambda_n}\}$  such that both  $V$  and  $V' = \{x_{\lambda_n} + x_{\chi_n}\}$  are  $\varepsilon$ -Kronecker sets in  $\mathbb{Q}$ , with the interpolation being done by characters  $g \in \widehat{\mathbb{Q}}$  such that  $g(\mathbb{Z}) = 1$ , and hence by characters on  $\mathbb{Q}/\mathbb{Z}$ . These can also be viewed as characters on  $C(p^\infty)$  if the domain is suitably restricted. Of course,  $g(\pi_\alpha(\lambda_n)) = g(x_{\lambda_n} + \mathbb{Z}) = g(x_{\lambda_n})$  and  $g(\pi_\alpha(\lambda_n + \chi_n)) = g(x_{\lambda_n} + x_{\chi_n})$ . It follows that both  $\{\pi_\alpha(\lambda_n)\}$  and  $\{\pi_\alpha(\lambda_n + \chi_n)\}$  are  $\varepsilon$ -Kronecker subsets of  $\Omega_\alpha$ .

The construction in the proof of the previous lemma also ensures that the pullbacks,  $E = \{\lambda_n\}$  and  $E' = \{\lambda_n + \chi_n\}$ , are disjoint, have distinct terms and are  $\varepsilon$ -Kronecker in  $\Gamma$ . That their union is not  $I_0$  follows immediately from Lemma 1.  $\square$

**REMARK 2.** *We note that similar arguments can be used to prove that under the assumption that  $\pi_\alpha(F)$  is infinite for some  $\alpha$  there are infinite sets,  $E, E'$ , that are  $I_0$  and have the property that for each  $\varepsilon > 0$  the sets  $E$  and  $E'$  contain cofinite subsets that are  $\varepsilon$ -Kronecker and whose union is not  $I_0$ . The latter statement is a consequence of the fact that the union of an  $I_0$  set and a finite set is known to be  $I_0$  (see [7, p.63]).*

### 3.3. Proof of the Theorem in Case 2.

**LEMMA 6.** *Suppose  $\pi_\alpha(F)$  is finite for all  $\alpha \in \mathcal{I}$ , but that  $\mathcal{I}_q$  is infinite for some  $q \geq 2$ , where*

$$\mathcal{I}_q = \{\alpha \in \mathcal{I} : \exists \lambda \in F \text{ s.t. } \pi_\alpha(\lambda) \text{ has order at least } q\}.$$

*Let  $|\exp(i\pi/q) - 1| = \varepsilon_q$ . There are infinite disjoint sets  $E \subset F$  and  $E' \subset F + F - F$  such that  $E$  and  $E'$  are both weak  $\varepsilon_q$ -Kronecker and  $I_0$ , but  $E \cup E'$  is not  $I_0$ .*

**PROOF.** Again, let  $H = \{\chi_n\}_{n=1}^\infty \subseteq (F - F) \setminus \{0\}$  be a countable set that clusters at 0.

First, assume  $\mathcal{I}_q$  is infinite for some  $q \geq 3$ . Let  $\beta_1 \in \mathcal{I}$  be chosen with the property that  $\pi_{\beta_1}(\chi_1) = 0$  and there is some  $\lambda_1 \in F$  with  $\pi_{\beta_1}(\lambda_1)$  having order  $\geq q$ . We can do this since there are only finitely many indices  $\beta$  with  $\pi_\beta(\chi_1) \neq 0$ . Now inductively choose  $\beta_n \in \mathcal{I}$  and  $\lambda_n \in F$  such that  $\pi_{\beta_n}(\lambda_n)$  has order at least  $q$ ,

$$\pi_{\beta_n}(\chi_m) = 0 \text{ for all } m \leq n \text{ and } \pi_{\beta_n}(\lambda_m) = 0 \text{ for all } m < n.$$

Set  $\Pi$  equal to the projection from  $\Omega$  onto  $\Lambda = \bigoplus \Omega_{\beta_n}$ . Then  $\{\Pi(\lambda_n)\}$  and  $\{\Pi(\lambda_n + \chi_n)\}$  are sequences of distinct elements of  $\Lambda$  such that  $\Pi(\lambda_m) \neq \Pi(\lambda_n + \chi_n)$  if  $m \neq n$ . Since  $\chi_n \neq 0$ , the sets  $E = \{\lambda_n\}$  and  $E' = \{\lambda_n + \chi_n\}$  are disjoint. Moreover  $\Pi(\lambda_n + \chi_n) = \pi_{\beta_n}(\lambda_n) + \rho_n$  where  $\pi_{\beta_k}(\rho_n) = 0$  for all  $k \geq n$ .

We claim that  $\{\Pi(\lambda_n + \chi_n)\}$  is weak  $\varepsilon_q$ -Kronecker. To prove this, note that a character  $g$  on  $\Lambda$  is specified as  $g = \{g_n\}$ , with each  $g_n$  a character on  $\Omega_{\beta_n}$ . Let  $M_n$  be the order of  $\pi_{\beta_n}(\lambda_n)$ . For any  $M_n$ -th root of unity,  $\omega_n$ , there is a character  $g_n$  such that  $g_n(\pi_{\beta_n}(\lambda_n)) = \omega_n$ . Thus given  $\{t_n\} \subseteq \mathbb{T}$ , we may inductively specify  $g_n$  so that

$$|g(\pi_{\beta_n}(\lambda_n)) - g(\rho_n)^{-1}t_n| \leq |\exp(i\pi/M_n) - 1| \leq \varepsilon_q.$$

Thus

$$|g(\Pi(\lambda_n + \chi_n)) - t_n| = |g(\pi_{\beta_n}(\lambda_n))g(\rho_n) - t_n| \leq \varepsilon_q,$$

which proves the claim.

The argument that  $\{\Pi(\lambda_n)\}$  is weak  $\varepsilon_q$ -Kronecker is similar. By Lemma 3,  $E$  and  $E'$  are also weak  $\varepsilon_q$ -Kronecker. As  $q \geq 3$ , we have  $\varepsilon_q \leq 1$  and hence  $E$  and  $E'$  are  $I_0$ . As before, their union is not  $I_0$ .

Otherwise, we can assume  $\mathcal{I}_2$ , but not  $\mathcal{I}_3$ , is infinite. Then there is a finite (possibly empty) set  $\mathcal{J} \subseteq \mathcal{I}$  such that for all  $\lambda \in F$  and all  $\alpha \in \mathcal{I} \setminus \mathcal{J}$ , either  $\pi_\alpha(\lambda) = 0$  or has order 2. Repeat the construction as above, but this time with the additional requirement that  $\beta_n \in \mathcal{I} \setminus \mathcal{J}$ . As before, the sets  $E, E'$  that arise from the construction are disjoint and they are both weak  $\sqrt{2}$ -Kronecker. Moreover,  $\Pi(E)$  and  $\Pi(E')$  are both independent sets of elements of order 2 and one can easily verify that such sets are  $I_0$  (c.f. [7, p. 66]). It follows from Lemma 3 that  $E$  and  $E'$  are both  $I_0$  sets, while their union is not.  $\square$

These three lemmas complete the proof of the main theorem since the assumption that  $F$  is infinite guarantees that either  $\pi_\alpha(F)$  is infinite for some  $\alpha$ , or there are infinitely many indices  $\alpha$  with  $\pi_\alpha(F)$  not trivial and in that case  $\mathcal{I}_q$  is infinite for some  $q \geq 2$ .

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DEPT. OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA  
N2L 3G1

*E-mail address:* `kehare@uwaterloo.ca`

DEPT. OF MATHEMATICS, UNIVERSITY OF HAWAII, HONOLULU, HI., USA 96822

*E-mail address:* `ramsey@math.hawaii.edu`